

THE SIGNIFICANCE OF VOIDS

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Received 1985 August 2; accepted 1985 October 23

ABSTRACT

The statistical significance of voids in the spatial distribution of galaxies or clusters of galaxies is studied. The probability per unit volume of finding a large void is expressed in terms of correlation functions. Numerical simulations are used to estimate the likelihood of observing a large void in the distribution of rich clusters of galaxies, assuming that rich clusters arose wherever suitably averaged primordial density fluctuations were unusually large.

Subject headings: cosmology — galaxies: clustering

I. INTRODUCTION

Although the homogeneity of the universe is often declared to be the fundamental principle of cosmology, the distribution of luminous matter in the universe is known to be highly nonuniform on relatively large distance scales. The very large scale nonuniformities in the spatial distribution of galaxies and clusters of galaxies are of great potential importance in cosmology. They provide observational information about the primordial energy density fluctuations at long wavelengths for which linear perturbation theory is valid; this information is therefore subject to a simple theoretical assessment.

Unfortunately, two-point (or N -point) correlation functions are typically so small at the relevant distances (tens or hundreds of megaparsecs) that they cannot be measured accurately. One might reasonably hope, however, to be able to deduce some of the crude features of the primordial spectrum of density fluctuations in the early universe by studying rare but dramatic features of the large-scale distribution of galaxies. Among such features are voids—large regions containing many fewer than the mean expected number of objects.

For example, in a magnitude-limited survey of galaxies, a region of diameter $60 h^{-1}$ Mpc has been observed (Kirshner *et al.* 1981) containing an unusually low number of galaxies. The sampled portion of this region was expected to contain 23 galaxies, but none were seen. Another example is an ellipsoidal void in the distribution of Abell clusters (with richness $R \geq 1$) that has a volume greater than $10^6 h^{-3}$ Mpc³ (Bachall and Soneira 1982). No rich clusters were observed in a region in which 10 were expected, after correcting for obscuration and other selection effects. Naively, the observation of large voids suggests that the primordial spectrum of the density fluctuations had significant power at long wavelengths. It is especially interesting to determine whether the observations are compatible with the scale-free spectrum (Harrison 1970; Peebles and Yu 1970; Zeldovich 1972) suggested by simple naturalness considerations.

This paper is a study of the significance of voids. In § II, the probability per unit volume of finding a void is expressed in terms of correlation functions. The main new result is a computation of the “entropy” factor, the number of effectively independent ways in which the empty region can be chosen. We conclude that the observed rich cluster void is not statistically significant; there is at least a 30% chance that a void of this size would occur even if the rich clusters were randomly distributed.

In § III, a numerical study of voids in the distribution of rich clusters is presented. In this analysis, it is assumed that the primordial density fluctuations obeyed Gaussian statistics with a power-law spectrum and that rich clusters arose wherever suitably averaged density fluctuations were unusually large (Kaiser 1984a). We compare the consequences of a power spectrum proportional to the wavenumber k (Harrison 1970; Peebles and Yu 1970; Zeldovich 1972) with those of a spectrum proportional to k^{-1} .

Section IV contains our conclusions.

II. VOIDS AS FLUCTUATIONS

To assess the significance of an observed void, we must estimate the probability that such a void would occur as a statistical fluctuation. For example, given some hypothetical form for the probability distribution of the primordial energy density perturbations, we should estimate the probability for a void of specified size and shape occurring somewhere in a catalog of objects like bright galaxies or rich clusters. If a void is seen for which the estimated probability is low, we can rule out the hypothetical distribution at some calculable statistical confidence level.

To illustrate this method, let us begin by considering voids in a sample of randomly distributed pointlike objects, without correlations. If the mean density of the objects is n , the probability of finding k objects in a particular region of arbitrary shape and volume V is given by the Poisson formula

$$P_k[V] = \frac{(nV)^k}{k!} e^{-nV}. \quad (1)$$

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To obtain equation (1), we divide the volume V into tiny cells of volume ΔV , take the probability for a cell to contain an object to be $n\Delta V$, and consider the cells to be independent. Then equation (1) is the limit of the binomial formula for k objects in $V/\Delta V$ cells, as $\Delta V \rightarrow 0$.

Now suppose that a void of volume V is seen in a sample distribution of objects with total volume V_{total} ; that is, the observed number k of objects in V is much less than the expected number nV . We wish to test the hypothesis that the objects are randomly distributed, without correlations, and must therefore compute the probability of such a void occurring as a statistical fluctuation. For V_{total} sufficiently large, the expected number of voids of a given size obviously scales linearly with V_{total} , so the quantity we wish to compute is $\mathbf{P}_k[V]$, the probability per unit total volume that there is some region of volume V which contains only k objects.

In fact, $\mathbf{P}_k[V]$ is ill defined unless we impose some restrictions, such as connectedness and convexity, on the shape of the volume V . Without such restrictions, there are always many ways of choosing a volume which avoids all the pointlike objects. Let us first consider the simplest case, the probability per unit total volume of a cubic void of fixed orientation.

If the objects are randomly distributed, without correlations, then the probability $P_k[V]$ that a cube with volume V and arbitrarily chosen center contains k objects is given by equation (1). There is an infinite number of ways of choosing the center of the cube, but obviously not all cubes can be considered independent trials; if a given cube contains k objects, then a cube displaced by a sufficiently small amount is likely also to contain k objects. Nonoverlapping cubes, on the other hand, *are* independent, but $\mathbf{P}_k[V]$ is actually much larger than $P_k[V]/V$ if $k \ll nV$. The reason is that if a void of volume V is a very rare event, then it is unlikely to be contained in a void of much larger volume. Therefore, we must position our trial cube very precisely in order to locate the void. If we cover the sample with of order V_{total}/V nonoverlapping trial cubes, the trial cubes will typically fail to detect a void even if one is present.

To understand the correct counting of effectively independent trial volumes, it is helpful to formulate a void-searching algorithm. Consider a set of cubes of volume V and fixed orientation whose centers form a fine lattice with spacing a and whose edges are aligned with the lattice. Then the mean number of cubes containing k objects is $P_k[V](V_{\text{total}}/a^3)$. But if a is small compared to the mean separation between objects, then the centers of the cubes that contain k objects will tend to form connected groups of lattice sites; the actual region containing only a given k objects is typically somewhat larger than V , so it is possible to jiggle the cube a bit without leaving that region. Each connected group of centers corresponds to a single void, which should only be counted once. The average number in a group $N_{\text{group}}(a)$ is proportional to a^{-3} , and the probability per unit total volume of finding a cubic void with volume V that contains only k objects is

$$\mathbf{P}_k[V] = \frac{P_k[V]}{a^3 N_{\text{group}}(a)}. \quad (2)$$

The denominator $a^3 N_{\text{group}}(a)$ in equation (2) is the mean volume of the region in which the center of the cube can be jiggled without the cube leaving the void; it has a limit V_{group} independent of a as $a \rightarrow 0$.

To compute V_{group} , we need to know the conditional probability, given a cube of volume V containing k objects, that a slightly displaced cube also contains only the same k objects, and thus is in the same group. For $k \ll nV$, this conditional probability is $P_0[\delta V]$, where δV is the additional volume contained in the displaced cube that is not contained in the original cube; that is the conditional probability is the probability that δV is empty. By integrating the conditional probability over the position of the center of the displaced cube, we obtain the mean volume of the group that contains the center of the original cube, averaged over all possible choices for the original cube from among those cubes containing k objects. But this quantity is *not* the same as V_{group} , because a randomly selected cube containing k objects is more likely to be a member of a large group than a small group. To avoid this bias, we must select a representative member of each group and calculate the mean value of the group volume by averaging over the representatives. In order to select a representative center from each group, we order the points of space according to

$$(x_1, y_1, z_1) < (x_2, y_2, z_2), \quad \text{if } \begin{cases} x_1 < x_2, \\ \text{or } x_1 = x_2, y_1 < y_2, \\ \text{or } x_1 = x_2, y_1 = y_2, z_1 < z_2, \end{cases} \quad (3)$$

and choose the representative to be the point of lowest value in the group $(x_{\min}, y_{\min}, z_{\min})$. For V sufficiently large, we then make a negligible error in assuming that the entire group lies in the octant $x > x_{\min}, y > y_{\min}, z > z_{\min}$ and may therefore restrict the integral over the displacement \mathbf{r} of the center of the cube to that octant. Note that having V large ensures that the integral over \mathbf{r} falls off sufficiently rapidly for large \mathbf{r} that the leading behavior of the integral is insensitive to how it is cut off. Furthermore, having $k \ll nV$ ensures that the likelihood of losing one of the original k in a shift by relevant \mathbf{r} is negligible. We thus obtain

$$V_{\text{group}} = \frac{1}{(n^3 V^2)}. \quad (4)$$

Plugging into equation (2), we obtain one of the main results of this section. For $k \ll nV$,

$$\mathbf{P}_k[V] = \frac{(nV)^3}{V} P_k[V]. \quad (5)$$

An important special case of equation (5) is the probability per unit total volume of finding an empty cube of fixed orientation and volume V ,

$$\mathbf{P}_0[V] = \frac{(nV)^3}{V} e^{-nV}. \quad (6)$$

The three powers of nV in equation (5) arise from integration over the three translational degrees of freedom of the trial cube in equation (4). The probability density for a spherical void of volume V has the same form (Politzer and Preskill 1986):

$$P_k[V] = \left(\frac{3\pi^2}{32}\right) \frac{(nV)^3}{V} P_k[V]. \quad (7)$$

If we consider more general shapes, the trial shapes have more degrees of freedom, and V_{group} is determined by an integral in a higher dimensional parameter space; more powers of nV result. For example, an ellipsoid of fixed volume has 8 degrees of freedom, three translational, three rotational, and two rescalings of the principle axes which leave the volume invariant. The probability per unit total volume of finding an ellipsoidal volume V containing k objects would have the form

$$P_k[V] = A \frac{(nV)^8}{V} P_k[V]. \quad (8)$$

For nV sufficiently large, A is a purely geometrical constant depending on the maximum accepted eccentricity.

As an application, we consider the statistical significance of the rich cluster void described in the introduction. If the expected number of objects is $nV = 10$, the probability per unit volume of a spherical void in a sample of randomly distributed objects is $P_0[V] \approx 0.04/V$, according to equation (7), and thus the probability of finding a spherical void in a sample of 70 objects is about 0.3. The observed void is actually highly ellipsoidal; finding such a void would not be at all improbable even if the rich clusters were randomly distributed. In a two-dimensional projection of the sky, the probability of seeing an empty elliptical region in which the expected number of objects is 10 is about 25%, according to a Monte Carlo simulation described in more detail in the next section.

Let us now consider how the previous discussion is modified if the positions of the objects are correlated. The probability $P_k[V]$ that an arbitrarily chosen volume contains k objects has been expressed in terms of correlation functions by White (1979). Using his result, we can easily find a corresponding expression for $P_k[V]$.

For completeness, we present a rederivation of White's formula for $P_k[V]$. To begin, we divide V_{total} into tiny cells of volume dV and associate with the i th cell a random variable n_i defined by

$$n_i = \begin{cases} 1 & \text{if } i\text{th cell contains an object} \\ 0 & \text{if } i\text{th cell is empty} \end{cases}, \quad (9)$$

Dimensionless connected k -point correlation functions $\xi_k(i_1, \dots, i_k)$ can be defined in terms of expectation values of strings of n_i . We have

$$\langle n_{i_1} \rangle = ndV \xi_1(i_1) = ndV, \quad (10)$$

where n is the mean density of objects; ξ_1 is defined to be one. The connected two-point function ξ_2 is defined by

$$\langle n_{i_1} n_{i_2} \rangle = (ndV)^2 [\xi_1(i_1)\xi_1(i_2) + \xi_2(i_1, i_2)]. \quad (11)$$

In general, we may write

$$\langle n_{i_1}, \dots, n_{i_N} \rangle = (ndV)^N \sum_{\{m_j\}} \sum_{\text{permutations}} (\xi_1^{m_1} \xi_2^{m_2} \xi_3^{m_3} \dots). \quad (12)$$

In equation (12) we sum over sets of nonnegative integers $\{m_j\}$ subject to the constraint

$$\sum_{j=1}^N j m_j = N. \quad (13)$$

The sum over permutations runs over all distinct ways of choosing the arguments of the ξ 's from among i_1, \dots, i_N ; there are $N! / [(1!)^{m_1} (2!)^{m_2} \dots (m_1!)(m_2!) \dots]$ such permutations.

Now consider the volume V consisting of the cells $i = 1, 2, \dots, M$. The probability that there are objects in the cells $i = 1, 2, \dots, k$ and none elsewhere in V is

$$\left\langle n_1, \dots, n_k \prod_{j=k+1}^M (1 - n_j) \right\rangle = \langle n_1, \dots, n_k \rangle - \sum_{j=k+1}^M \langle n_1, \dots, n_k n_j \rangle + \sum_{j_1 < j_2 = k+1}^M \langle n_1, \dots, n_k n_{j_1} n_{j_2} \rangle. \quad (14)$$

To find $P_k[V]$, we must sum over all possible choices of the k occupied cells, obtaining

$$P_k[V] = \frac{1}{k!} \left[\sum'_{i_1, \dots, i_k=1}^M \langle n_{i_1}, \dots, n_{i_k} \rangle - \frac{1}{1!} \sum'_{i_1, \dots, i_{k+1}=1}^M \langle n_{i_1}, \dots, n_{i_{k+1}} \rangle + \frac{1}{2!} \sum'_{i_1, \dots, i_{k+2}=1}^M \langle n_{i_1}, \dots, n_{i_{k+2}} \rangle + \dots \right]; \quad (15)$$

here the prime indicates that the summation variables are required to take distinct values.

It is convenient to express $P_k[V]$ in terms of $P_0[V]$. Since, according to equation (12), $\langle n_{i_1}, \dots, n_{i_N} \rangle$ is formally proportional to n^N , we see that

$$P_k[V] = \frac{(-n)^k}{k!} \frac{\partial^k}{\partial n^k} P_0[V], \quad (16)$$

where the differentiation is carried out with the dimensionless correlation functions held fixed, and

$$P_0[V] = \sum_{N=0}^{\infty} \frac{(-1)^N}{N!} \sum'_{i_1, \dots, i_N=1}^M \langle n_{i_1}, \dots, n_{i_N} \rangle. \quad (17)$$

Invoking the decomposition of equation (12) and replacing sums by integrals, we find

$$\sum'_{i_1, \dots, i_N=1}^M \langle n_{i_1}, \dots, n_{i_N} \rangle = n^N \sum_{\{m_j\}} \frac{N!}{[(1!)^{m_1} (2!)^{m_2} \dots (m_1!) (m_2!) \dots]} (I_1^{m_1} I_2^{m_2} \dots), \quad (18)$$

where

$$I_j[V] = \int_V d^3x_1 \dots \int_V d^3x_j \xi_j(\mathbf{x}_1, \dots, \mathbf{x}_j). \quad (19)$$

The summation over N in equation (17) can now be performed: summing over N removes the constraint of equation (13) on the m_j , and the expression for P_0 factorizes

$$\begin{aligned} P_0[V] &= \sum_{\{m_j\}} \prod_{j=1}^{\infty} \frac{1}{(m_j)!} \left[\frac{(-n)^j}{j!} I_j \right]^{m_j} \\ &= \exp \left[\sum_{j=1}^{\infty} \frac{(-n)^j}{j!} \int_V d^3x_1 \dots \int_V d^3x_j \xi_j(\mathbf{x}_1, \dots, \mathbf{x}_j) \right]. \end{aligned} \quad (20)$$

Equation (20) is an expansion of $\ln P_0[V]$ in powers of the density n of objects, analogous to the cluster expansion of statistical mechanics. Loosely speaking, it is a good expansion if the typical separation between objects is large compared to the characteristic length scale at which correlations are strong. Equation (20) and equation (16) were previously derived by White (1979). When all ξ_j , ($j \geq 2$) vanish, equations (16) and (20) reduce to equation (1).

The translational invariance of the ξ_j allows one power of V to be extracted from each of the integrals in equation (20). If correlation functions decay rapidly at large distances, then the remaining integrals approach constant values for sufficiently large V . Therefore,

$$P_0[V] \xrightarrow{V \rightarrow \infty} \exp [-nVa^{-1}(n)], \quad (21)$$

where $a^{-1}(n)$ is independent of V .

For randomly distributed objects, the probability that V is empty is given by the Poisson formula $P_0[V] = \exp [-nV]$. Equation (21) has the same volume dependence as the Poisson formula, but with a renormalized number density

$$\begin{aligned} n_{\text{cluster}} &= na^{-1}(n) \\ &= - \sum_{j=1}^{\infty} \frac{(-n)^j}{j!} \int d^3x_2 \dots \int d^3x_j \xi_j(\mathbf{0}, \mathbf{x}_2, \dots, \mathbf{x}_j). \end{aligned} \quad (22)$$

Thus it is natural to interpret equation (21) in terms of a "cluster model." In this model, the correlations cause the objects to be grouped together into clusters, small compared to V , with a mean number a of objects per cluster. If the clusters themselves are randomly distributed, then equation (21) is the probability that V contains no clusters.

Now, by the same logic applied previously to the case of randomly distributed objects, we can express the probability per unit total volume $\mathbf{P}_0[V]$ of finding a spherical region V that contains no objects in terms of $P_0[V]$,

$$\mathbf{P}_0[V] = P_0[V]/V_{\text{group}}[V]. \quad (23)$$

To compute V_{group} , we must, as before, know the conditional probability, given a sphere of volume V containing no objects, that a slightly displaced sphere also contains no objects. This conditional probability is

$$P_0[V \cup \delta V]/P_0[V] = \exp \left\{ \sum_{j=1}^{\infty} \frac{(-n)^j}{j!} \left[\int_{V \cup \delta V} d^3x_1 \dots \int_{V \cup \delta V} d^3x_j \xi_j(\mathbf{x}_1, \dots, \mathbf{x}_j) - \int_V d^3x_1, \dots, \int_V d^3x_j \xi_j(\mathbf{x}_1, \dots, \mathbf{x}_j) \right] \right\}, \quad (24)$$

where δV is the additional volume in the displaced sphere that is not contained in the original sphere.

For asymptotically large spherical volumes V , we need only consider small values of the displacement r , and it becomes sensible to expand the argument of the exponential in equation (24) in powers of r , retaining only the leading term. The result is

$$P_0[V \cup \delta V]/P_0[V] = \exp [-n'_{\text{cluster}} \delta V], \quad (25)$$

where

$$n'_{\text{cluster}} = - \sum_{j=1}^{\infty} \frac{(-n)^j}{(j-1)!} \int_V d^3x_2 \dots \int_V d^3x_j \xi_j(\mathbf{x}_1, \dots, \mathbf{x}_j). \quad (26)$$

In equation (26), \mathbf{x}_1 is a fixed point on the edge of the spherical volume. Equation (25) implies that the group volume in the correlated case is the same as the uncorrelated case except that n is replaced by n'_{cluster} . Therefore,

$$\mathbf{P}_0[V] = \frac{3\pi^2}{32} \frac{(n'_{\text{cluster}} V)^3}{V} P_0[V]. \quad (27)$$

An identical formula relates \mathbf{P}_k and P_k for k small. Equation (27) applies, in general, in the large V limit. Furthermore, if the correlation functions decay sufficiently rapidly, n'_{cluster} approaches a constant independent of V as $V \rightarrow \infty$. But note that equation (26) for n'_{cluster} is not the same as the expression equation (22) for n_{cluster} , except in the special case in which $\xi_j = 0$ for $j \geq 3$.

Provided the length scales characterizing the asymptotic behavior of the correlation functions are sufficiently small, an analogous argument can be applied to any region bounded by a smooth surface. That is, a formula analogous to equation (27) will apply to any region with a sufficiently small surface to volume ratio, whose boundary has a sufficiently small mean square curvature.

Before closing this section, we should add a caveat concerning the cluster model interpretation of equation (21). For fixed $k \neq 0$, the behavior of $P_k[V]$ as $V \rightarrow \infty$ given by equation (16) and (20) cannot typically be interpreted in terms of randomly distributed clusters. The leading term in $P_k[V]$ in the large volume limit is

$$P_k[V] = \frac{(-n)^k}{k!} \frac{\partial^k}{\partial n^k} P_0[V] \xrightarrow{V \rightarrow \infty} \frac{(bnV)^k}{k!} \exp(-a^{-1}nV), \quad (28)$$

where

$$b = a^{-1} \left(1 + n \frac{\partial}{\partial n} \ln a^{-1} \right). \quad (29)$$

On the other hand, the Poisson formula for $P_k[V]$, if the objects are grouped in randomly distributed clusters with mean number a per cluster, is

$$P_k[V] = \frac{1}{(k/a)!} (a^{-1}nV)^{k/a} \exp(-a^{-1}nV). \quad (30)$$

Equations (28) and (30) do not agree for $a \neq 1$ and $k \neq 0$; in fact, they scale differently with V .

Although equation (28) cannot be sensibly interpreted in terms of randomly distributed clusters, it is useful; it characterizes the large V behavior of $P_k[V]$ in terms of only two parameters, a and b . Thus, the results of numerical calculations of $P_k[V]$ for large volumes, like those we describe in § III, can be reported as fits to a and b , provided that correlation functions fall off rapidly enough at large distances for equation (21) to be valid.

There are special cases in which an interpretation of $P_k[V]$ in terms of randomly distributed clusters does make sense. After all, a distribution of objects consisting of independent clusters with uncorrelated positions is a possible distribution and will have $P_k[V]$ given by the Poisson formula for large V . In such cases, equation (28) is not valid because the leading term for $V \rightarrow \infty$ which we have kept is actually zero.

Consider, for example, randomly distributed clusters which always consist of two objects with vanishing separation. In this case, the connected two-point correlation function is

$$\xi_2(\mathbf{x}_1, \mathbf{x}_2) = \frac{1}{n} \delta^3(\mathbf{x}_1 - \mathbf{x}_2), \quad (31)$$

and all higher correlation functions vanish. Plugging into equation (29), we find that b vanishes. (Recall that $\partial/\partial n$ is to be evaluated with ξ_2 held fixed.) In fact, P_k given by equations (16) and (20) agrees exactly with the Poisson formula (eq., [30], with $a = 2$) for k even and vanishes for k odd.

Next, consider a distribution with connected two-point function

$$\xi_2(\mathbf{x}_1, \mathbf{x}_2) = \frac{(a-1)}{n} \delta(\mathbf{x}_1 - \mathbf{x}_2) \quad (32)$$

and vanishing higher correlation functions. We first note that consistency requires $a \leq 2$. Otherwise some of the P_k (e.g., $k = 1$) will be negative. It does not make sense for the two-point function to require clusters of more than two objects if there are no higher correlations. For $a < 2$, equation (28) applies, and an interpretation of $P_k[V]$ in terms of randomly distributed clusters is not applicable.

The above example seems representative of the general case; distributions to which the cluster model can be applied to $P_k[V]$, $k \neq 0$, are not typical. An important special case, however, is that in which all connected correlation functions integrate to zero in the large volume limit

$$0 = \int d^3x_2, \dots, d^3x_j \xi_j(0, \mathbf{x}_2, \dots, \mathbf{x}_j). \quad (33)$$

In that case, $P_k[V]$ is given for large V by the Poisson formula for randomly distributed *single* objects, in spite of the apparent short-range clustering implied by the nonzero ξ_j terms.

III. RICH CLUSTER VOIDS

Our objective is to infer properties of the primordial energy density fluctuations from the existence of voids. For this purpose, voids of the largest possible size are of the greatest interest; the larger the void, the greater our confidence that it is a feature which continues to evolve today in a manner well described by linear perturbation theory, rather than in a complicated nonlinear way.

Therefore, we will mostly confine our attention to voids in the distribution of rich clusters of galaxies. Because of their large intrinsic brightness, rich clusters can be seen at great distances, and huge rich cluster voids are potentially observable. Also, the positions of rich clusters are known to be much more strongly correlated than the positions of galaxies (Bachall and Soneira 1983; Klypin and Kopylov 1983; Hauser and Peebles 1973); the two-point correlation function for Abell clusters (richness $R \geq 1$) is unity at about $25 h^{-1}$ Mpc, while the two-point correlation function for galaxies is unity at $5 h^{-1}$ Mpc. Thus, huge rich cluster voids are

more likely than huge galaxy voids. Although we have noted that the rich cluster void already observed does not have much statistical significance, we anticipate the accumulation of more dramatic data as the sample of rich clusters with measured redshift increases.

An explanation for the enhanced correlations of rich clusters was recently proposed by Kaiser (1984a). If the primordial energy density fluctuations are Gaussian and rich clusters formed wherever the fluctuations, averaged over an appropriate volume, are unusually large, then the rich cluster two-point function is enhanced relative to the energy density two-point function. In fact, all the higher rich cluster correlation functions can be computed in terms of the spectrum of the underlying energy density fluctuations (Politzer and Wise 1984). Kaiser's model of rich clusters seems reasonable if galaxies and clusters formed by a hierarchical mechanism, rather than by the fragmentation of larger objects. We assume its validity in our analysis.

In this section, we briefly review the calculation of the rich cluster correlation functions in Kaiser's model and consider whether the results of § II can be used to compute the likelihood of a rich cluster void. Finding analytic methods inadequate, we resort to numerical simulations to assess the significance of voids.

To begin, we derive the consequences of Kaiser's model for the rich cluster correlation functions. If $\epsilon_{av}(\mathbf{x})$ denotes the value of an energy density fluctuation $\delta\rho(\mathbf{x})/\langle\rho\rangle$ averaged over a region of radius R centered at \mathbf{x} , then Gaussian statistics implies that the probability distribution for $\epsilon_{av}(\mathbf{x})$ at each (smeared) point \mathbf{x} is

$$P_0(\epsilon_{av}) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\epsilon_{av}^2/2\sigma^2}, \quad (34)$$

where

$$\sigma^2 = \int \frac{d^3k}{(2\pi)^3} \xi_\rho(\mathbf{k}) e^{-k^2 R^2/2}. \quad (35)$$

Here $\xi_\rho(\mathbf{k})$ is the power spectrum of the energy density fluctuations, and the momentum integral is cut off for $|\mathbf{k}| \geq R^{-1}$. The two-point correlation function for ϵ_{av} is

$$\langle \epsilon_{av}(\mathbf{x}_1) \epsilon_{av}(\mathbf{x}_2) \rangle = \xi_\rho(\mathbf{x}_1 - \mathbf{x}_2) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k} \cdot (\mathbf{x}_1 - \mathbf{x}_2)} \xi_\rho(\mathbf{k}) e^{-k^2 R^2/2}. \quad (36)$$

and all higher connected correlation functions vanish.

In Kaiser's model, a rich cluster forms in a region of suitably chosen size R if ϵ_{av} exceeds a threshold value t . Thus, in the notation of equation (9), the random variable $n_{rc}(\mathbf{x})$, averaged over many trial distributions, can be represented by

$$n_{rc}(\mathbf{x}) = \mathcal{G}[\epsilon_{av}(\mathbf{x}) - t] dV. \quad (37)$$

The asymptotic predictions for the rich cluster correlation functions, for $t \gg \sigma$ and large separation, are insensitive to the form of the threshold in equation (37). Therefore, for ease of calculation, we may smooth out the threshold and write (Kaiser 1984b)

$$n_{rc}(\mathbf{x}) = \exp [t\epsilon_{av}(\mathbf{x})/\sigma^2] dV, \quad (38)$$

whose expectations are also sharply peaked at $\epsilon_{av} \approx t$ for $t \gg \sigma$. Thus, we obtain (Politzer and Wise 1984)

$$\begin{aligned} (n_{rc} dV)^{-N} \langle n_{rc}(\mathbf{x}_1) \cdots n_{rc}(\mathbf{x}_N) \rangle &\approx \frac{\langle \exp [t\epsilon_{av}(\mathbf{x}_1)/\sigma^2] \cdots \exp [t\epsilon_{av}(\mathbf{x}_N)/\sigma^2] \rangle}{\langle \exp [t\epsilon_{av}(\mathbf{x})/\sigma^2] \rangle^N} \\ &= \exp \left[(t/\sigma)^2 \sum_{i < j} \frac{\xi_\rho(\mathbf{x}_i - \mathbf{x}_j)}{\xi_\rho(\mathbf{0})} \right], \end{aligned} \quad (39)$$

where Gaussian statistics have been invoked to evaluate the expectation value, and n_{rc} is the mean density of rich clusters.

Equation (39) is valid for $t \gg \sigma$ and $\xi_\rho(\mathbf{x}_i - \mathbf{x}_j) \ll \xi_\rho(\mathbf{0})$. We see that "rare events" with threshold $t \gg \sigma$ have enhanced two-point correlations

$$\xi_{rc,2}(\mathbf{x}_1 - \mathbf{x}_2) = \exp [t^2/\sigma^4 \xi_\rho(\mathbf{x}_1 - \mathbf{x}_2)] - 1, \quad (40)$$

and that the enhancement increases as t does. This result is in accord with the observation that Abell clusters have correlations enhanced compared to those of galaxies, and that rarer clusters of higher "richness" have stronger correlations than less rich clusters. Equation (39) also allows us to express all higher rich cluster correlation functions in terms of the two-point function, without explicit reference to the correlations of the underlying density fluctuations,

$$\langle n_{rc}(\mathbf{x}_1) \cdots n_{rc}(\mathbf{x}_N) \rangle \approx (n_{rc} dV)^N \prod_{i < j} [1 + \xi_{rc,2}(\mathbf{x}_i - \mathbf{x}_j)]. \quad (41)$$

In particular, the rich clusters have nontrivial three-point correlations which may be measurable.

Equation (40) relates the rich cluster two-point correlation function to the two-point correlation function of the underlying Gaussian primordial energy density perturbations. The enhanced rich cluster correlations can be measured at relatively large distances, and the form of the energy density correlations in this distance range can thus be inferred, assuming the validity of Kaiser's model and linear perturbation theory. In order to probe the energy density fluctuation spectrum at still longer wavelengths for which rich cluster two-point correlations are not measurable, we wish to derive, in Kaiser's model, an expression for the probability of a large rich cluster void. This probability, as we noted in § II, is sensitive to the higher correlations of the rich clusters, as well.

Equation (41), combined with equation (20), enables us to express the probability $P_0[V]$ that a specified volume V contains no rich clusters in terms of the rich cluster two-point function as

$$P_0[V] = \exp \left[-n_{\text{rc}}(\odot) + \frac{1}{2} n_{\text{rc}}^2(\odot\text{---}\odot) - \frac{1}{6} n_{\text{rc}}^3(\odot\odot\odot + \odot\odot\odot + \odot\odot\odot + \odot\odot\odot) + \cdots \right]. \quad (42)$$

Here, in an obvious notation,

$$\begin{aligned} \odot &= \int_V d^3x_1 = V, \\ \odot\text{---}\odot &= \int_V d^3x_2 \int_V d^3x_1 \xi_{\text{rc},2}(\mathbf{x}_1, \mathbf{x}_2), \\ \odot\odot\odot &= \int_V d^3x_1 \int_V d^3x_2 \int_V d^3x_3 \xi_{\text{rc},2}(\mathbf{x}_1, \mathbf{x}_2) \xi_{\text{rc},2}(\mathbf{x}_2, \mathbf{x}_3), \\ &\text{etc.} \end{aligned} \quad (43)$$

In writing equation (42), we implicitly assume that the integrals of correlation functions in equation (20) receive a dominant contribution from the region in which the asymptotic formula equation (41) applies.

For the power spectrum of the Gaussian energy density fluctuations, we consider in detail two expressions which approach power laws for long wavelengths

$$\bar{\xi}_p(\mathbf{k}) \propto |\mathbf{k}|^p, \quad p = 1, -1. \quad (44)$$

The $p = 1$ case is the natural scale-free spectrum proposed by Harrison (1970), Peebles and Yu (1970), and Zeldovich (1972). The $p = -1$ case is an alternative spectrum, chosen (Kaiser 1984a) to fit the reported measurements of the rich cluster two-point correlation function in the distance range $10\text{--}100 h^{-1} \text{ Mpc}$.

At large distances, the rich cluster two-point function $\xi_{\text{rc},2}$ given by equation (40) approaches the form of the energy density two-point correlation function, but enhanced by the factor t^2/σ^4 . Thus, if the energy density fluctuations have the $p = 1$ scale-free spectrum, then $|\xi_{\text{rc},2}(r)|$ decays like r^{-4} for large r . This decay is sufficiently rapid for the discussion of § II to apply; $P_0[V]$ approaches $\exp[-n_{\text{cluster}} V]$ as $V \rightarrow \infty$, with n_{cluster} independent of V .

Unfortunately, n_{cluster} is not readily calculable. The problem is that the “tree” graphs in equation (42) and the “loop” graphs are sensitive to the form of the correlation functions at short distances, which are not small. For example

$$\frac{n_{\text{rc}}^2}{6V} \odot\odot \xrightarrow{V \rightarrow \infty} \frac{1}{6} n_{\text{rc}}^2 \int \frac{d^3k}{(2\pi)^3} [\bar{\xi}_{\text{rc},2}(\mathbf{k}) e^{-k^2 R^2/2}]^3 \quad (45)$$

gives a contribution to n_{cluster} enhanced by powers of R^{-1} which is potentially important, and there are graphs of higher order which also depend sensitively on the ultraviolet cutoff R^{-1} . Since equation (41) does not apply at short distances comparable to R , and since many graphs contribute anyway, we are unable to calculate accurately. The sensitive dependence of n_{cluster} on the behavior of the correlation functions at short distances indicates a strong tendency for the rich clusters to themselves form close clusters with a size comparable to our averaging radius R .

If the energy density fluctuations have the $p = -1$ spectrum, then equations (21), (22), (28), and (29) of § II are not applicable, because the decay of the rich cluster correlation functions at large distances is not sufficiently rapid. The rich cluster two-point function behaves like

$$\xi_{\text{rc},2}(\mathbf{r}') \approx (r_0/r')^2 \quad (46)$$

for large r' , and thus, for example

$$\begin{aligned} \frac{n_{\text{rc}}}{2V} \odot\text{---}\odot &= \frac{1}{2} n_{\text{rc}} \int_V d^3r' \xi_{\text{rc},2}(\mathbf{r}') \\ &\approx 2\pi (n_{\text{rc}} r_0^3) (r/r_0), \end{aligned} \quad (47)$$

if the region of volume V is a sphere of radius r . We see that, for sufficiently large r , the expansion in equation (42) will fail to converge well; $P_0[V]$ cannot be reliably calculated, even though the integrals of the correlation functions are dominated by the asymptotic long distance region in which equation (41) applies. Empirically, $n_{\text{rc}} r_0^3 \approx 0.1$ for the rich clusters, so high-order tree graphs can be important for r comparable to the typical separation between clusters. There are also graphs which are ultraviolet-sensitive, quantifying the tendency of the rich clusters to themselves form closely grouped clusters.

Because our general formulas cannot be usefully applied to the problem of estimating the probability of a large rich cluster void in the cases of interest, we have resorted to numerical methods. We generated sample distributions of rich clusters by first generating sample energy density fluctuation fields $\epsilon_{\text{av}}(\mathbf{x})$ and then identifying the (smeared) points above threshold as rich clusters. The sample fields $\epsilon_{\text{av}}(\mathbf{x})$ were chosen by a simulated Gaussian stochastic process with two-point correlation function

$$\bar{\xi}_p(\mathbf{k}) = \frac{(2\pi)^3 \epsilon_H^2 k^p}{H_0^4} A^2(k). \quad (48)$$

Here ϵ_H is a constant which fixes the normalization of the power spectrum of the fluctuations, and $H_0 = 100 h \text{ km s}^{-1} \text{ Mpc}$ is the present value of the Hubble constant. The factor $A(k)$ arises from the logarithmic growth of fluctuations in the dark matter (assumed to be cold) on scales inside the horizon before the time of matter domination; it can be computed in linear perturbation theory. If the cold dark matter has critical density $\Omega = 1$, then an approximate expression for A is

$$A = (1 + 0.04x_m^2 + 0.0005x_m^4)^{-1}, \quad (49)$$

with

$$x_m = \left(\frac{2}{3}\right)^{2/3} k t_0^{2/3} t_m^{1/3}, \quad (50)$$

where t_0 is the present time and t_m is the time of matter domination. Using an expansion rate of the universe appropriate for the case of three massless neutrinos, one estimates

$$x_m \approx 11h^{-2}k_{\text{Mpc}^{-1}}, \quad (51)$$

where $k_{\text{Mpc}^{-1}}$ is the wavenumber measured in Mpc^{-1} . The approximation equation (49) is reasonable for $x_m \leq 8$ but seriously underestimates $A(k)$ for $x_m > 8$.

To generate the $\epsilon_{\text{av}}(\mathbf{x})$ of a sample universe, we define $\epsilon_{\text{av}}(\mathbf{x})$ on an $N \times N \times N$ cubic lattice with periodic boundary conditions and randomly choose its Fourier coefficients according to the Gaussian distribution

$$P[\tilde{\epsilon}_{\text{av}}(\mathbf{k})] \propto \exp \left[-\frac{1}{2V} \tilde{\epsilon}_{\rho}(\mathbf{k}) |\tilde{\epsilon}_{\text{av}}(\mathbf{k})|_2 e^{-k^2 R^2/2} \right], \quad (52)$$

where V is the volume of the simulated universe and R is the radius of the region over which the energy density fluctuations are to be averaged. A sample $\tilde{\epsilon}_{\text{av}}(\mathbf{k})$ is generated by the Box-Muller method (Dahlquist and Björck 1974), using a linear congruential random number generator (Knuth 1969), and then a sample $\epsilon_{\text{av}}(\mathbf{x})$ is obtained via a fast Fourier transform.

For each sample $\epsilon_{\text{av}}(\mathbf{x})$, the positions of the rich clusters are identified with the points above threshold in $\epsilon_{\text{av}}(\mathbf{x})$. To perform this identification, we must fix the parameters of Kaiser's model. After determining the spectrum by selecting values for p , ϵ_H , h , and Ω , we retain the freedom to choose the averaging radius R and the threshold t . These parameters are chosen so that the mean density of rich clusters is $n_{\text{rc}} \approx (50 h^{-1} \text{ Mpc})^{-3}$, and the rich cluster two-point correlation function is of order one at a separation of $25 h^{-1} \text{ Mpc}$. As discussed by Kaiser, we should also demand that roughly 2%–5% of the total mass resides in the rich clusters, and that the protoclusters have recollapsed by the present time.

A beautiful feature of Kaiser's model is that it enables us to study the distribution of clusters without ever considering nonlinear gravitational dynamics. But the cost of this simplicity is some ambiguity. Frequently $\epsilon_{\text{av}}(\mathbf{x})$ is found to be above the threshold at two neighboring sites with a separation comparable to the averaging radius R ; a more detailed model is needed to determine whether these two sites should be counted as two distinct clusters or as a single cluster. It is important to do this counting correctly. If we frequently count two rich clusters where there should actually be one, we will overestimate the tendency of the rich clusters to cluster together; since the mean number density of rich clusters is fixed, we will hence overestimate the probability of a void. To avoid overcounting, a modified version of Kaiser's procedure is used. We identify rich clusters only with local maxima of $\epsilon_{\text{av}}(\mathbf{x})$ that are above threshold.

All our runs were performed on a 64^3 cubic lattice with periodic boundary conditions and with lattice spacing $8 h^{-1} \text{ Mpc}$. Each sample universe is thus $512 h^{-1} \text{ Mpc}$ across, but because of artificial effects introduced by the periodic boundary conditions, the reliably simulated universe is somewhat smaller. The averaging radius is chosen to be $R = 8 h^{-1} \text{ Mpc}$.

By generating many sample universes, and counting the number of rich clusters in randomly placed spheres, we can measure $P_k[V]$ for spherical volumes. More subtle is the measurement of $P_0[V]$, the probability per unit total volume of finding any convex region of volume V which contains no rich clusters. To search for large voids, we made pictures of the rich cluster configurations and then scanned them by eye. The pictures were designed to mimic the actual observations; the rich clusters contained in a hemisphere of radius $160 h^{-1} \text{ Mpc}$ were projected onto the surface of the hemisphere, which was then projected onto a plane by an area-preserving (sinusoidal) map. The projections contained a mean number of 68 rich clusters.

Since we demand that the voids be visible in projection, we are liable to miss some of the voids present in the configurations. Also, since identifying rich clusters with peaks of $\epsilon_{\text{av}}(\mathbf{x})$ defined on a lattice with spacing $R = 8 h^{-1} \text{ Mpc}$ removes close pairs, we are underestimating the likelihood of a void. We believe, then, that our void search procedure provides a lower bound on the probability of a large void.

For the $p = 1$ scale-free spectrum, we choose parameters $\epsilon_H = 3.4 \times 10^{-6}$, $h = \frac{1}{2}$, $\Omega = 1$, $t = 1.95$, and $R = 8 h^{-1} \text{ Mpc}$. The chosen value of ϵ_H saturates the upper bound derived from the dipole anisotropy of the microwave background for an $\Omega = 1$ universe dominated by cold dark matter (Abbott and Wise 1984). We found that values of the Hubble constant $h \leq \frac{1}{2}$ are required in order that the rich cluster two-point correlation function remain positive out to distances greater than $25 h^{-1} \text{ Mpc}$. The energy density two-point function, and hence also the rich cluster two-point function, must eventually change sign, because $\int d^3x \tilde{\epsilon}_{\rho}(\mathbf{x}) = \tilde{\epsilon}_{\rho}(k=0) = 0$. The value of h is crucial, because it determines the location of the break in the spectrum parameterized by $A(k)$ in equation (49). Thus, Kaiser's model, the $p = 1$ spectrum, and the measured rich cluster correlation functions might be shown to be incompatible, if the measurement of h improves, or positive rich cluster correlations are firmly established at distances much beyond $25 h^{-1} \text{ Mpc}$. For our choice of parameters, the rich cluster two-point correlation function is unity at about $15 h^{-1} \text{ Mpc}$.

We measured the distribution $P_k[V]$ for spheres with $n_{rc} V = 17$ and 29. (The case $n_{rc} V = 17$ is shown in figure 1a.) The asymptotic large V formula equation (28) was found to apply, to the small k tail, with $a = 1.10$ and $b = 0.90$. Thus $P_0[V]$ has a cluster model interpretation, with a number of objects per cluster of 1.10.

For the $p = -1$ spectrum, we chose parameters $\epsilon_H = 8.4 \times 10^{-5} H_0$, $h = \frac{1}{2}$, $\Omega = 1$, $t = 1.63$, and $R = 8 h^{-1}$ Mpc. The chosen value of ϵ_H is slightly below the upper bound derived from the dipole anisotropy of the microwave background for an $\Omega = 1$ universe dominated by cold dark matter. For this spectrum, positivity of the rich cluster two-point correlation function does not constrain h ; since $\int d^3x \xi_p(x) = \xi_p(k=0) = \infty$, the correlation function need not change sign. We found $\xi_{rc,2}$ is unity at about $24 h^{-1}$ Mpc, with $h = \frac{1}{2}$.

The measured $P_k[V]$ is plotted in Figure 1b, for spheres with $n_{rc} V = 17$. We do not expect, nor do we find, that the asymptotic large V formula equation (28) applies to the small k tail of this distribution.

Searching for voids, we scanned the sample rich cluster configurations one by one. For each sample hemisphere, we measured the area of the largest circle in the projected sky that contained no rich clusters. The size of the largest circular void seen in each hemisphere can be specified by N , the mean number of rich clusters expected in a region of this area. The measured probability distribution for N is shown in Figure 2 for three cases—randomly distributed rich clusters, rich clusters in Kaiser's model with the $p = 1$ spectrum, and Kaiser's model with the $p = -1$ spectrum. In all three cases, the mean number of rich clusters in the sky was chosen to be $N_{total} = 68$, corresponding to a density $n_{rc} \approx (50 h^{-1} \text{ Mpc})^{-3}$, and 100 sample configurations were generated.

For randomly distributed objects, we also measured the area of the largest rectangle in the projected sky which contained no rich clusters. A rectangular void is more likely than a circular void of the same area, because a rectangle has additional orientation and shape degrees of freedom not shared by a circle. We found that the integral of the tail of the distribution for N , in the case of rectangular voids, is fitted reasonably well by

$$V_{total} P_0[V] \approx 0.1 N^4 e^{-N} \frac{N_{total}}{N}. \quad (53)$$

Equation (53) is the asymptotic large N formula, analogous to equation (7), for the probability of an empty rectangular region with expected number of objects at least N in a sample of N_{total} randomly distributed objects in two dimensions. Of 100 rich cluster configurations generated, roughly 25% contained an empty rectangular region, visible in projection, with $N \geq 10$. About 1% contained an empty rectangular region with $N \geq 14$. Two typical projections are shown in Figure 3. On the sole basis of the $N = 10$

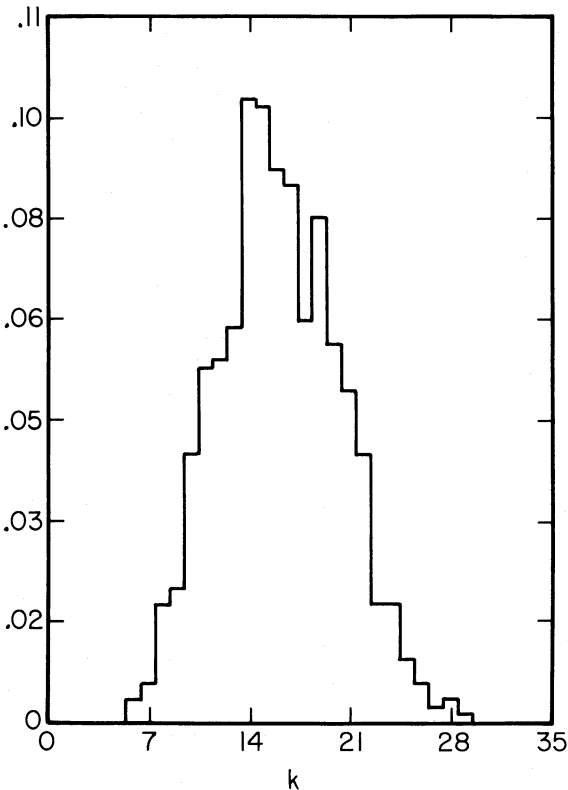


FIG. 1a

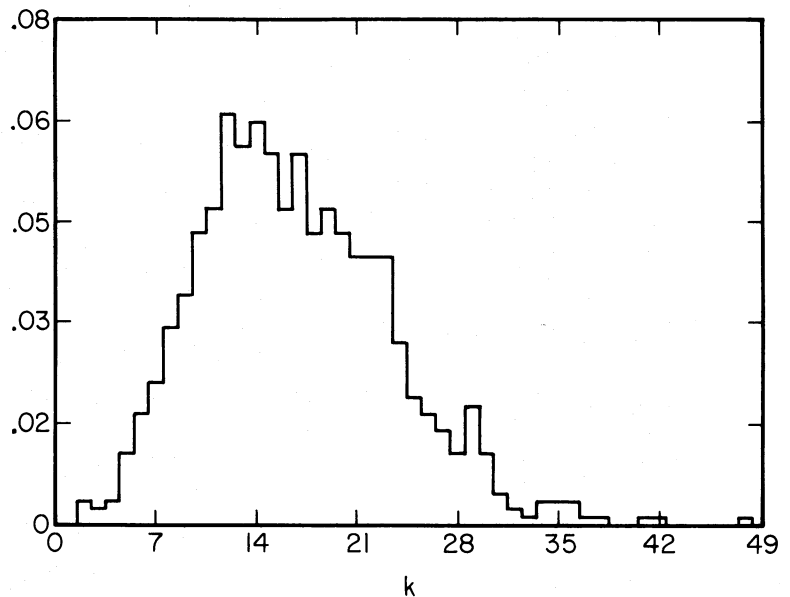


FIG. 1b

FIG. 1.—The probability $P_k[V]$ that a randomly selected sphere of volume V contains k rich clusters, for $n_{rc} V = 17$, in two cases: (a) Kaiser's model with $p = 1$ spectrum; (b) Kaiser's model with $p = -1$ spectrum.

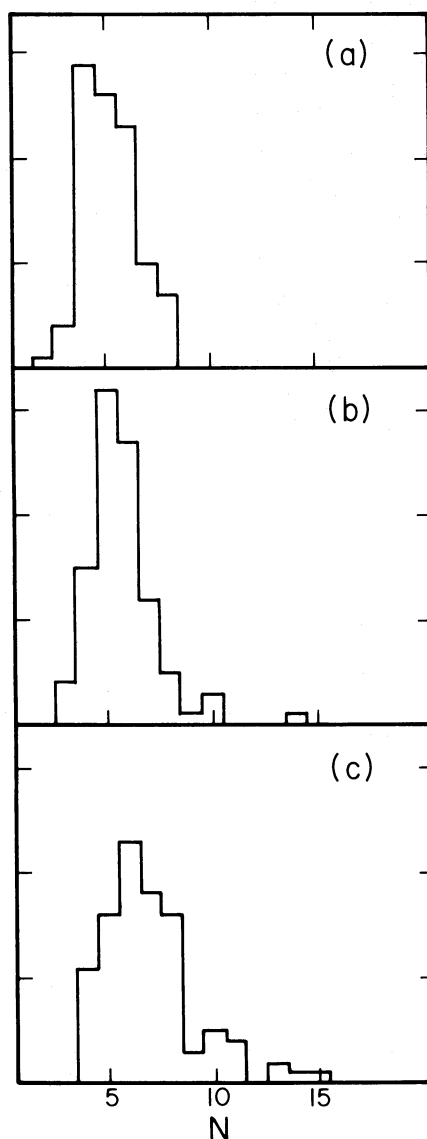


FIG. 2.—Probability $P[N]$ that the expected number of rich clusters is N in the largest visible circular void in a sample projected hemisphere. Based on 100 simulations with mean $N_{\text{total}} = 68$ in each of the three cases (a) randomly distributed clusters, (b) Kaiser's model with $p = 1$ spectrum and (c) Kaiser's model with $p = -1$ spectrum.

void observed by Bachall and Soneira, the hypothesis that the rich clusters are randomly distributed cannot be ruled out with better than 75% confidence.

For rich clusters distributed according to Kaiser's model with a scale-free $p = 1$ spectrum of underlying density fluctuations, we found a distribution for the largest circular void not dramatically different from the corresponding distribution for randomly distributed objects. The distribution in Figure 2b roughly matches that in Figure 2a if N is rescaled, $N \rightarrow 0.9N$, as suggested by the cluster model.

For the Kaiser model with a $p = -1$ spectrum, large voids are more common. For example, 32% of the sample projections contained a circular void in which the expected number of rich clusters was eight or more, as opposed to 11% for the $p = 1$ spectrum and 7% for the random case.

In Kaiser's model with the underlying Gaussian density fluctuations obeying a $p = 1$ spectrum, rich clusters tend to cluster together, making voids more likely than if the rich clusters were randomly distributed. But we find that this effect is small and not easily discerned. In Kaiser's model with a $p = -1$ spectrum, the excessive power at long wavelengths makes voids more likely. Figures 4 and 5 show respectively a typical projected hemisphere and one with a large void for the $p = 1$ spectrum. Figures 6 and 7 show, respectively, a typical projected hemisphere and one with a large void for the $p = -1$ spectrum. For the hemisphere with 68 rich clusters considered in our simulations, we find that the $p = -1$ spectrum is not so readily distinguished from the $p = 1$

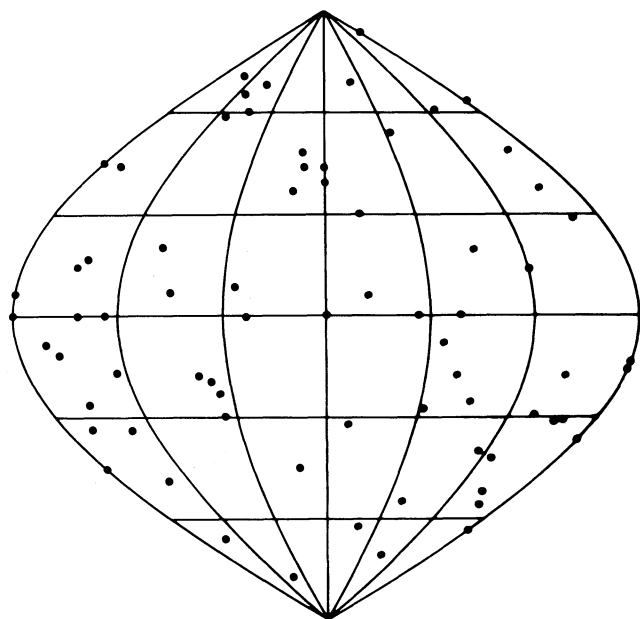


FIG. 3a

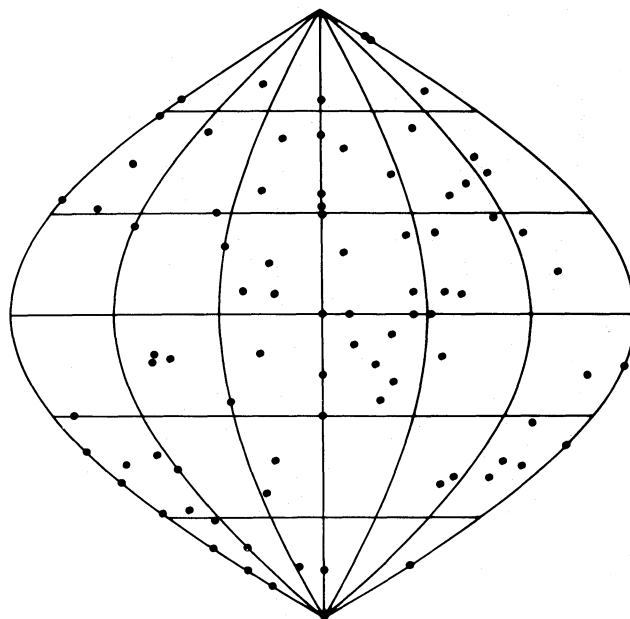


FIG. 3b

FIG. 3.—Two typical projected hemispheres; randomly distributed objects

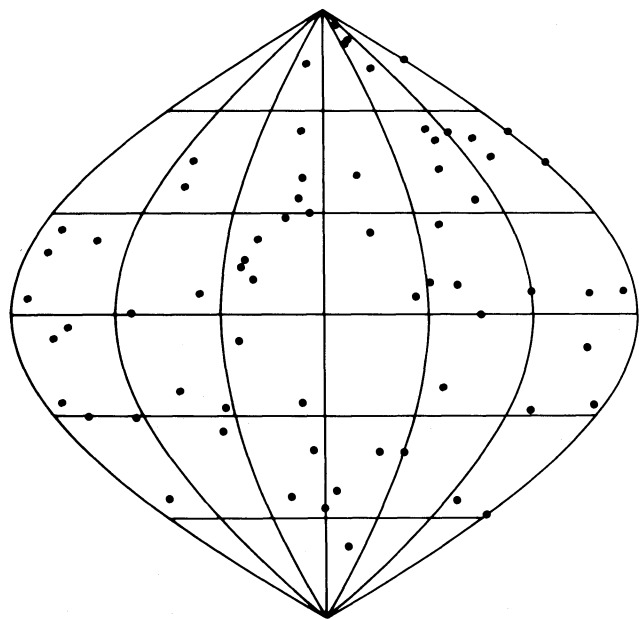


FIG. 4

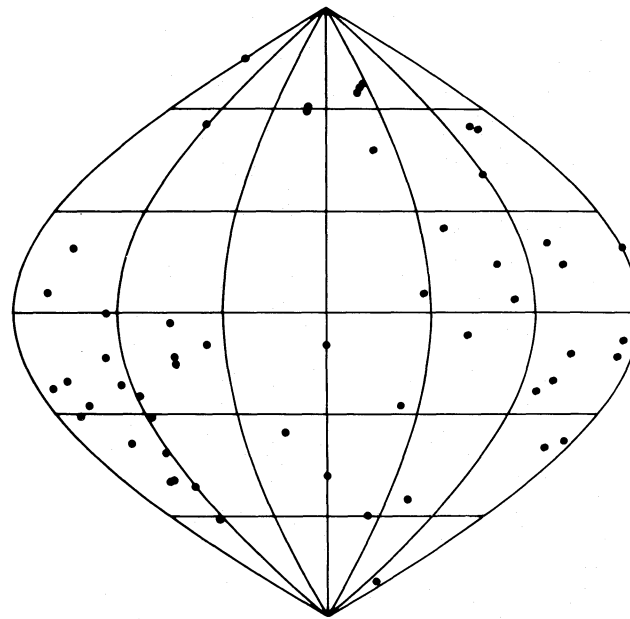


FIG. 5

FIG. 4.—A typical projected hemisphere; Kaiser's model with $p = 1$ spectrum
FIG. 5.—A projection with a large void; Kaiser's model with $p = 1$ spectrum

spectrum, on the basis of the likelihood of a large circular void visible in projection. The distributions in Figure 2 reflect the large void counts in a sample of the two-dimensional projections of 100 such hemispheres. Although the distributions appear to be distinguishable, we have not explored the dependence of these distributions on the uncertainties in the simulation. However the simulation suggests that three-dimensional data concerning the distribution of a large sample of rich clusters might well provide statistically significant tests of hypotheses about the primordial energy density fluctuation spectrum.

IV. CONCLUSIONS

The large-scale structure of the universe evolved from small primordial energy density fluctuations. It is of great interest to

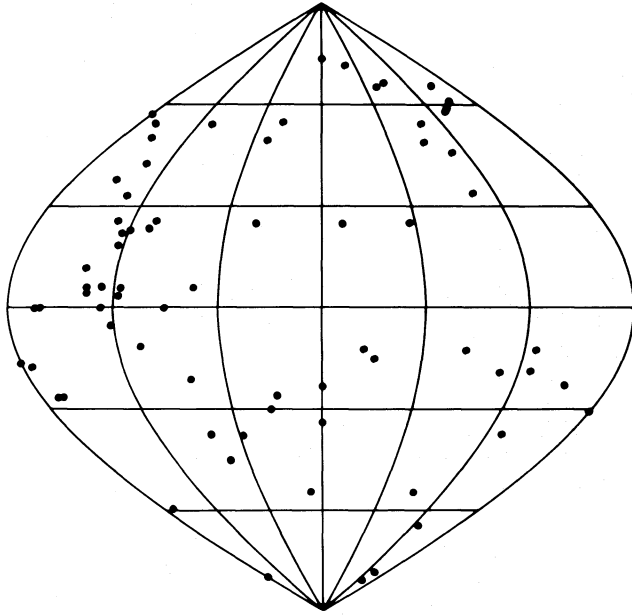


FIG. 6

FIG. 6.—A typical projected hemisphere; Kaiser's model with $p = -1$ spectrum

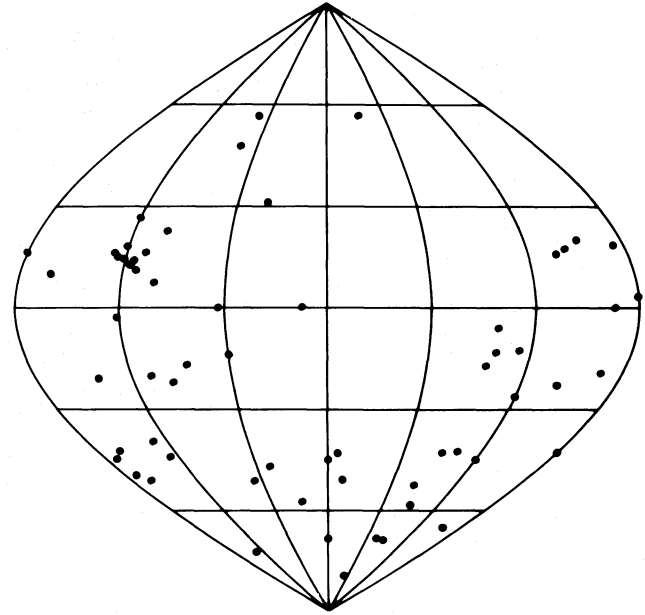


FIG. 7

FIG. 7.—A projection with a large void; Kaiser's model with $p = -1$ spectrum

determine, by observation, the nature of these energy density fluctuations. We have suggested that voids, particularly large voids in the distribution of rich clusters of galaxies, may be useful for this purpose.

We have concluded that an observed rich cluster void is not inconsistent with the hypothesis that the rich clusters are randomly distributed. However, statistically significant tests of hypotheses concerning the primordial energy density fluctuations may become possible as the sample of rich clusters with measured redshift increases.

Our analysis is based on Kaiser's interpretation of rich clusters as rare events in the primordial Gaussian density fluctuations; it thus presupposes a hierarchical clustering, or cold dark matter, scenario. This interpretation does not apply to a pancake picture in which clusters formed by the fragmentation of larger structures.

We are particularly interested in testing the hypothesis that the primordial fluctuations have a $p = 1$ spectrum, since this hypothesis follows from simple and general naturalness requirements. Because gravitational instabilities cause the density fluctuations to grow like τ^2 , where τ is conformal time, we write, for wavelengths which entered the horizon after the time of matter domination,

$$\tilde{\epsilon}(\mathbf{k}) = \hat{a}(\mathbf{k})\tau^2. \quad (54)$$

Here $\hat{a}(\mathbf{k})$ is a random variable with the canonical dimension of distance. If the process which generated the density fluctuations did not introduce any astrophysically relevant distance scale, then a simple scaling relation can be derived by naive dimensional analysis,

$$\langle \hat{a}(\lambda \mathbf{k}_1) \cdots \hat{a}(\lambda \mathbf{k}_n) \rangle = \lambda^{-n} \langle \hat{a}(\mathbf{k}_1) \cdots \hat{a}(\mathbf{k}_n) \rangle. \quad (55)$$

A special case of equation (55) is

$$\langle \hat{a}(\mathbf{k}_1) \hat{a}(\mathbf{k}_2) \rangle \propto |\mathbf{k}_1| \delta^3(\mathbf{k}_1 + \mathbf{k}_2); \quad (56)$$

the two-point function has the Harrison-Zeldovich spectrum. Equation (55) generalizes the Harrison (1970), Zeldovich (1972), Peebles and Yu (1970) notion of naturalness to the higher energy density correlation functions. From it follows the scaling law

$$P_R(z) = \left(\frac{R'}{R} \right)^2 P_R \left(\frac{R'}{R} z \right) \quad (57)$$

for the probability distribution $P_R(z)$ of

$$z = \frac{1}{V} \int_V d^3x \epsilon(x), \quad (58)$$

the average value of ϵ in a region with linear dimension R and volume V . Equation (57) is satisfied by any scale-free distribution of density fluctuations, whether Gaussian or not.

The naturalness criterion equation (55), together with the homogeneity and isotropy of space, determines the form of the

two-point correlation function up to normalization but allows many different types of (non-Gaussian) higher correlations. Although density fluctuations with Gaussian statistics are predicted in the new inflationary universe scenario (Guth 1981; Albrecht and Steinhardt 1982; Linde 1982), we feel that, in general, the motivation for Gaussian statistics is not as strong as the motivation for a scale-free distribution. Thus, if the observed distribution of rich clusters were found to be incompatible with the interpretation of rich clusters as rare events in Gaussian noise with a $p = 1$ spectrum, we would be at least as inclined to modify the statistics as to modify the spectrum.

We have mainly confined our attention here to the distribution of rich clusters, but it is also very interesting to consider whether large-scale irregularities in the distribution of bright galaxies can be explained by a model in which galaxies are regarded as rare events in Gaussian noise. Such a model has great appeal, because it offers to explain why light is more clumped than mass in the universe, and thus to reconcile virial studies suggesting $\Omega \approx 0.2$ with theoretical prejudices in favor of $\Omega \approx 1$. An analysis of galaxy voids based on Kaiser's model might therefore be valuable.

We thank S. Coleman, R. P. Feynman and J. Salmon for helpful comments. The work was supported in part by the US Department of Energy under contracts DEAC-03-81-WR40050 and DE-AT03-38-ER31118. M. B. W. is supported in part by the DOE Outstanding Junior Investigator program under contract No. DE-GF03-84ER40172.

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